Solution to Homework Assignment No. 3

1. (a) Let $\boldsymbol{V}_a$ be the subspcae whose vectors have equal components; then $\boldsymbol{V}_a = \{(v_1, v_1, v_1, v_1) : v_1 \in \mathcal{R}\}$. Since all vectors in $\boldsymbol{V}_a$ have equal components, we can use $(1,1,1,1)$ to span the subspace $\boldsymbol{V}_a$. Therefore, a basis can be given by

$$(1,1,1,1)\,.$$

(b) Let $\boldsymbol{V}_b$ be the subspace that all vectors in $\boldsymbol{V}_b$ whose components add to zero; then $\boldsymbol{V}_b = \{(a,b,c,d) : a+b+c+d = 0, a,b,c,d \in \mathcal{R}\}$. And we can obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

Then we have to find the nullspace of $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, and we can observe that columns 2, 3 and 4 are free columns. Thus the special solutions are given by

$$\begin{aligned} (a,b,c,d) &= (-1,1,0,0) \\ (a,b,c,d) &= (-1,0,1,0) \\ (a,b,c,d) &= (-1,0,0,1)\,. \end{aligned}$$

Since the special solutions obtained are independent and span the nullspace, they form a basis. Therefore, we have a basis:

$$(-1,1,0,0)\,,(-1,0,1,0)\,,(-1,0,0,1)\,.$$

(c) Let $\boldsymbol{V}_c$ be the subspace whose vectors are perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$, i.e.,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $(a,b,c,d) \in \boldsymbol{V}_c$, and $a,b,c,d \in \mathcal{R}$. Then we perform Gaussian elimination to find the reduced row echelon form:

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} &\Longrightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \\ &\Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \end{aligned}$$

Thus we have two pivots and two free variables, and we obtain

$$\begin{cases} a = -c - d \\ b = c + d. \end{cases}$$

Substitute $(c, d) = (1, 0)$, and $(c, d) = (0, 1)$ into the equations above, and we can obtain the special solutions:

$$(a, b, c, d) = (-1, 1, 1, 0)$$
$$(a, b, c, d) = (-1, 1, 0, 1).$$

Therefore, a basis can be given by

$$(-1, 1, 1, 0), (-1, 1, 0, 1).$$

(d) We know that $\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By the definition of the column space, we have

$$\boldsymbol{C}(\boldsymbol{I}) = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} : x_1, x_2, x_3, x_4 \in \mathcal{R} \right\}.$$

Therefore, a basis is given by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

To find the nullspace of $\boldsymbol{I}$, since $\boldsymbol{I}\boldsymbol{x} = \boldsymbol{x}$ for every vector $\boldsymbol{x}$, we have $\boldsymbol{x} = \boldsymbol{0}$. Therefore, a basis for $\boldsymbol{N}(\boldsymbol{I})$ is the empty set.

2. To find a basis for $\boldsymbol{S}$, we have

$$\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

And we can observe that b, c, d are free variables. Therefore, we have special solutions

$$(0, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)$$

which form a basis for $\boldsymbol{S}$. To find a basis for $\boldsymbol{T}$, we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have two pivots and two free variables, and the special solutions are

$$(-1, 1, 0, 0), (0, 0, 2, 1)$$

which form a basis for $\boldsymbol{T}$. To find a basis for $\boldsymbol{S} \cap \boldsymbol{T}$, we have

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Perform Gaussian elimination, and we can obtain:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$
$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Since we have three pivots and one free variable, the special solution is

$$(-3, 3, 2, 1)\,.$$

which is a basis for $\boldsymbol{S} \cap \boldsymbol{T}$. Therefore, the dimension of $\boldsymbol{S} \cap \boldsymbol{T}$ is 1.

3.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \boldsymbol{LU}.$$

(a) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{r_1} \\ \boldsymbol{a}_{r_2} \\ \boldsymbol{a}_{r_3} \end{bmatrix}$ where $\boldsymbol{a}_{r_1}, \boldsymbol{a}_{r_2}, \boldsymbol{a}_{r_3}$ are row vectors of $\boldsymbol{A}$. Then we can have

$$\begin{aligned} \boldsymbol{a}_{r_1} &= 1 \cdot (1\ 2\ 3\ 4) \\ \boldsymbol{a}_{r_2} &= 6 \cdot (1\ 2\ 3\ 4) + 1 \cdot (0\ 1\ 2\ 3) \\ \boldsymbol{a}_{r_3} &= 9 \cdot (1\ 2\ 3\ 4) + 8 \cdot (0\ 1\ 2\ 3) + 1 \cdot (0\ 0\ 1\ 2) \end{aligned}$$

and also observe that the row space is

$$\boldsymbol{C}(\boldsymbol{A}^T) = \{a\,(1\ 2\ 3\ 4) + b\,(0\ 1\ 2\ 3) + c\,(0\ 0\ 1\ 2) : a, b, c \in \mathcal{R}\}.$$

Since (1 2 3 4), (0 1 2 3), and (0 0 1 2) are independent, a basis for $\boldsymbol{C}(\boldsymbol{A}^T)$ can be given by

$$(1\ 2\ 3\ 4)\,, (0\ 1\ 2\ 3)\,, (0\ 0\ 1\ 2)\,.$$

(b) Let $\boldsymbol{A} = [\boldsymbol{a}_{c_1}\ \boldsymbol{a}_{c_2}\ \boldsymbol{a}_{c_3}\ \boldsymbol{a}_{c_4}]$ where $\boldsymbol{a}_{c_1}$, $\boldsymbol{a}_{c_2}$, $\boldsymbol{a}_{c_3}$ and $\boldsymbol{a}_{c_4}$ are column vectors of

$\boldsymbol{A}$. Then we can have

$$\begin{aligned}
\boldsymbol{a}_{c_1} &= 1 \cdot \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\boldsymbol{a}_{c_2} &= 2 \cdot \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\boldsymbol{a}_{c_3} &= 3 \cdot \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\boldsymbol{a}_{c_4} &= 4 \cdot \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

and we can observe that the column space is

$$\boldsymbol{C}(\boldsymbol{A}) = \left\{ a \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : a, b, c \in \mathcal{R} \right\}.$$

Since $\begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are independent, a basis for $\boldsymbol{C}(\boldsymbol{A})$ can be given by

$$\begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) To find the nullspace of $\boldsymbol{A}$, we transform the upper-triangular matrix $\boldsymbol{U}$ to the reduced row echelon form $\boldsymbol{R}$ as follows:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} &\Longrightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
&\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{aligned}$$

Thus we can find the special solution:

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

Hence a basis for $\boldsymbol{N}(\boldsymbol{A})$ can by given by

$$\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}.$$

(d) Note that $\boldsymbol{A}$ is a 3×4 matrix. Since the dimension of $\boldsymbol{C}(\boldsymbol{A})$ is 3, the dimension of the left nullspace $\boldsymbol{N}(\boldsymbol{A}^T)$ is $3-3=0$. Hence $\boldsymbol{N}(\boldsymbol{A}^T)=\{\boldsymbol{0}\}$, and a basis for $\boldsymbol{N}(\boldsymbol{A}^T)$ is the empty set.

4. (a) To find a basis for the row space of $\boldsymbol{B}$:
By observation, we can find that the rows are replications of rows 1 and 2. Therefore, we can obtain

$$\boldsymbol{C}(\boldsymbol{B}^T)=\{x\,(1\ 0\ 1\ 0\ 1\ 0\ 1\ 0)+y\,(0\ 1\ 0\ 1\ 0\ 1\ 0\ 1):x,y\in\mathcal{R}\}$$

and a basis for $\boldsymbol{C}(\boldsymbol{B}^T)$ can be given by

$$(1\ 0\ 1\ 0\ 1\ 0\ 1\ 0)\,,(0\ 1\ 0\ 1\ 0\ 1\ 0\ 1)\,.$$

Therefore, the rank of $\boldsymbol{B}$ is 2.

(b) To find a basis for the left nullspace of $\boldsymbol{B}$:
Since $\boldsymbol{B}$ is symmetric, $\boldsymbol{B}^T=\boldsymbol{B}$, and rows 3 to 8 are replications of rows 1 and 2, we can have

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\boldsymbol{x}=[x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8]^T$ which satisfies $\boldsymbol{B}^T\boldsymbol{x}=\boldsymbol{0}$. Then we obtain

$$\begin{cases} x_1=-x_3-x_5-x_7 \\ x_2=-x_4-x_6-x_8. \end{cases}$$

Therefore, a basis for $\boldsymbol{N}(\boldsymbol{B}^T)$ can be obtained from the special solutions as given by

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(c) To find a basis for the row space of $\boldsymbol{C}$:
Since rows 7 and 8 are identical to rows 1 and 2, respectively, and the numbers r, n, b, q, k, p are all different, we can find that the row space of $\boldsymbol{C}$ is

$$\begin{aligned}\boldsymbol{C}(\boldsymbol{C}^T) &= \{a\,[r\ n\ b\ q\ k\ b\ n\ r]+b\,[p\ p\ p\ p\ p\ p\ p\ p]:a,b\in\mathcal{R}\} \\ &= \{a\,[r\ n\ b\ q\ k\ b\ n\ r]+b'\,[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]:a,b'\in\mathcal{R}\}.\end{aligned}$$

Therefore, a basis for $\boldsymbol{C}(\boldsymbol{C}^T)$ can be given by

$$[r\ n\ b\ q\ k\ b\ n\ r]\,,[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]\,.$$

The rank of $\boldsymbol{C}$ is 2. (Note that we assume $p \neq 0$.)

(d) To find a basis for the left nullspace of $\boldsymbol{C}$:
Since rows 6, 7, 8 of $\boldsymbol{C}^T$ are identical to rows 3, 2, 1, respectively, we only have to consider rows 1 to 5 in $\boldsymbol{C}^T$. Let $\boldsymbol{y} = [y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7\ y_8]^T$ be a vector in the left nullspace of $\boldsymbol{C}$, i.e., $\boldsymbol{C}^T\boldsymbol{y} = \boldsymbol{0}$. Since rows 3 to 5 can be reduced to the all-zero row and the numbers r, n, b, q, k, p are all different, we can have

$$\begin{bmatrix} r & p & 0 & 0 & 0 & 0 & p & r \\ n & p & 0 & 0 & 0 & 0 & p & n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then a basis for $\boldsymbol{N}(\boldsymbol{C}^T)$ can be obtained from the special solutions as given by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(e) To find a basis for the nullspace of $\boldsymbol{C}$:
Since rows 7, 8 of $\boldsymbol{C}$ are identical to rows 2, 1, respectively, and rows 3 to 6 are all-zero rows, we only have to consider rows 1 and 2 in $\boldsymbol{C}$. Let $\boldsymbol{z} = [z_1\ z_2\ z_3\ z_4\ z_5\ z_6\ z_7\ z_8]^T$ be a vector in the nullspace of $\boldsymbol{C}$, i.e., $\boldsymbol{C}\boldsymbol{z} = \boldsymbol{0}$.

Then we can have

$$\begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \quad \frac{1}{r-n} \begin{bmatrix} r-n & 0 & b-n & q-n & k-n & b-n & 0 & r-n \\ 0 & r-n & r-b & r-q & r-k & r-b & r-n & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that we assume $r \neq 0$ and $p \neq 0$. Then a basis can be obtained from the special solutions as given by

$$\begin{bmatrix} n-b \\ b-r \\ r-n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} n-q \\ q-r \\ 0 \\ r-n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} n-k \\ k-r \\ 0 \\ 0 \\ r-n \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} n-b \\ b-r \\ 0 \\ 0 \\ 0 \\ r-n \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

5. We can have

$$\boldsymbol{C}(\boldsymbol{A}^T) = \{(a,\ -a) : a \in \mathcal{R}\}$$

and

$$\boldsymbol{N}(\boldsymbol{A}) = \{(b,\ b) : b \in \mathcal{R}\}.$$

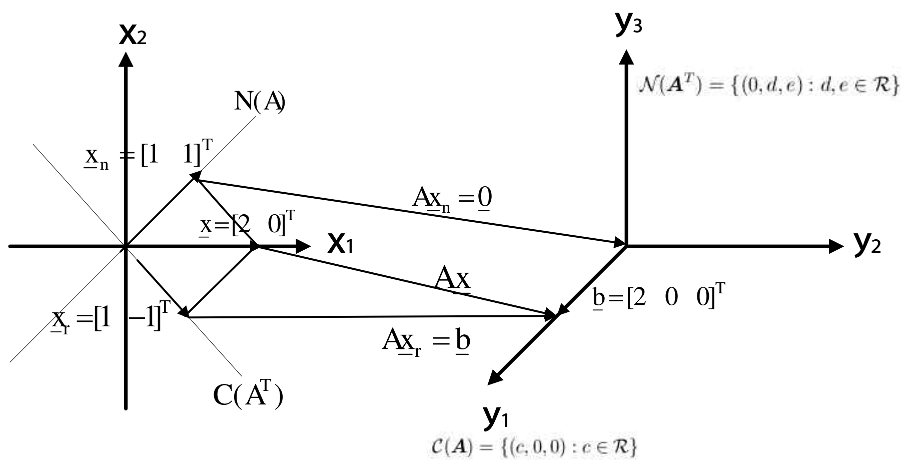
Therefore,

$$\boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boldsymbol{x}_r + \boldsymbol{x}_n$$

where

$$\boldsymbol{x}_r = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \boldsymbol{x}_n = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For this example, Figure 4.3 can be redrawn as



6. (a) Since the inner product of the zero vector and any other vector is always zero, we know that $\boldsymbol{S}^{\perp} = \mathcal{R}^3$.

(b) Let a vector in $\boldsymbol{S}^{\perp}$ be (x_1, x_2, x_3), where $x_1, x_2, x_3 \in \mathcal{R}$. Then we have

$$\begin{aligned} & (x_1, x_2, x_3) \cdot (1, 1, 1) = x_1 + x_2 + x_3 = 0 \\ \Longrightarrow \quad & x_3 = -x_1 - x_2 \end{aligned}$$

Therefore, the subspace $\boldsymbol{S}^{\perp}$ is

$$\boldsymbol{S}^{\perp} = \{x_1\,(1, 0, -1) + x_2\,(0, 1, -1) : x_1, x_2 \in \mathcal{R}\}.$$

(c) Let a vector in $\boldsymbol{S}^{\perp}$ be (y_1, y_2, y_3), where $y_1,\ y_2,\ y_3 \in \mathcal{R}$. Then we have

$$\begin{cases} (y_1, y_2, y_3) \cdot (1, 1, 1) = y_1 + y_2 + y_3 = 0 \\ (y_1, y_2, y_3) \cdot (1, 1, -1) = y_1 + y_2 - y_3 = 0 \end{cases}$$

$$\Longrightarrow \quad \begin{cases} y_2 = -y_1 \\ y_3 = 0 \end{cases}$$

Therefore, the subspace $\boldsymbol{S}^{\perp}$ is

$$\boldsymbol{S}^{\perp} = \{y_1\,(1, -1, 0) : y_1 \in \mathcal{R}\}$$

and a basis is

$$(1, -1, 0)\,.$$

7. We have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The projection of $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of $\boldsymbol{A}$ is then

$$\boldsymbol{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

The projection matrix $\boldsymbol{P}$ is a 4×4 square matrix given by

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. (a) We can find two vectors in the plane $x - y - 2z = 0$ as

$$\boldsymbol{v}_1 = (1, 1, 0)\,, \ \boldsymbol{v}_2 = (2, 0, 1)\,.$$

Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\boldsymbol{A}^T \boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Its inverse can be found as

$$\left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}.$$

Therefore, we can obtain the projection matrix as

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{A}\left(\boldsymbol{A}^T \boldsymbol{A}\right)^{-1} \boldsymbol{A}^T \\ &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}. \end{aligned}$$

(b) From the plane equation $x - y - 2z = 0$, we know that

$$\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

We can then have $\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ which is perpendicular to the plane. Thus we obtain

$$\boldsymbol{Q} = \frac{\boldsymbol{e}\boldsymbol{e}^T}{\boldsymbol{e}^T\boldsymbol{e}} = \frac{\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}}{1 + 1 + 4} = \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}.$$

Therefore, the projection matrix is given by

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{Q} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

which is identical to the result in (a).